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Integrable spin chains associated with $\widehat{sl_q(n)}$ and $\widehat{sl_{p,q}(n)}$

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Abstract. The Hopf structure of the central extension of the $U_q(\widehat{sl(n)})$ algebra is considered. The intertwine matrix induces new integrable spin chain models. We show the relation of these models and the biparametric spin chain $\widehat{sl_{p,q}(n)}$ models. The cases $n = 2$ are $n = 3$ are discussed, and for $n = 2$ we obtain the model of Dasgupta and Chowdhury. The case $n = 3$ is solved by the nested Bethe ansatz method, and the dependence of the Bethe equations on the second parameter introduced is demonstrated.

1. Introduction

The search for integrable models is an important problem and has deserved great attention over the two last decades. The isotropic and anisotropic spin chains of Heisenberg occupy a central position in such studies. The mathematical structure arising from these relatively simple models is astonishingly rich. The Yang–Baxter equation (YBE), based on the original treatment of Baxter and the quantum inverse formulation due to Faddeev and collaborators [1] are the key to finding new solvable models.

The quantum groups [2] constitute an elegant formalism for obtaining, in a consistent mathematical way, objects fulfilling the YBE and therefore, providing new solvable spin chains [3].

In this paper, we find a set of integrable models by considering a central extension of the algebra $U_q(\widehat{sl\,(n)})$, which obviously introduces a suitable definition of the coproduct on its Hopf algebra. The models so obtained are related to those derived from the coloured braid group representations [4], and they are the two-parameter deformed quantum groups $\widehat{sl_{p,q}\,(n)}$ [5, 6].

The present paper is organized as follows. In the next section we develop the formalism and show the relations with other models in the cases $n = 2$ and 3.

In third section, the model with $n = 3$ is solved by the nested Bethe ansatz (NBA) method. The Bethe equations obtained, show the dependence on the second parameter that will introduced a new degree of freedom in its solutions compared with that obtained with $\widehat{sl_p\,(n)}$ [7, 8].

2. Formulation

Consider $U_q(\widehat{sl\,(n)})$, the universal covering of the affine algebra, and let its generators be $\{E_i, F_i, H_i\}_{i=0}^{n-1}$. This algebra has a central extension $\{Z\}$ whose elements are a multiples of the identity, $Z = \lambda I$, λ being a parameter. Then, if q is not a root of unity, the elements of the fundamental representation will be characterized by two parameters, λ and the affine parameter of the algebra x [9].

The $U_q(\widehat{sl(n)})$ with central extension has a Hopf algebra. The coproduct is not uniquely determinate, so we can define one coproduct Δ

$$\Delta(E_i) = ZK_i \otimes E_i + E_i \otimes K_i^{-1} \qquad i = 1, \ldots, n-1 \tag{2.1a}$$

$$\Delta(F_i) = Z^{-1}k_i \otimes F_i + F_i \otimes K_i^{-1} \qquad i = 1, \ldots, n-1 \tag{2.1b}$$

$$\Delta(E_0) = Z^{-(n-1)}K_0 \otimes E_0 + E_0 \otimes K_0^{-1} \tag{2.1c}$$

$$\Delta(F_0) = Z^{n-1}K_0 \otimes F_0 + F_0 \otimes K_0^{-1} \tag{2.1d}$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \qquad i = 0, \ldots, n-1 \tag{2.1e}$$

$$\Delta(Z) = Z \otimes Z \tag{2.1f}$$

and its symmetric coproduct Δ'

$$\Delta'(E_i) = K_i^{-1} \otimes E_i + E_i \otimes ZK_i \tag{2.2a}$$

$$\Delta'(F_i) = K_i^{-1} \otimes F_i + F_i \otimes Z^{-1}K_i^{-1} \tag{2.2b}$$

$$\Delta'(E_0) = K_0^{-1} \otimes E_0 + E_0 \otimes Z^{-(n-1)}K_0^{-1} \tag{2.2c}$$

$$\Delta'(F_0) = K_0^{-1} \otimes F_0 + F_0 \otimes Z^{n-1}K_0^{-1} \tag{2.2d}$$

$$\Delta'(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \tag{2.2e}$$

$$\Delta'(Z) = Z \otimes Z. \tag{2.2f}$$

Both coproducts must be related by a transformation matrix $R_{1,2}$ such that

$$R_{1,2} \cdot \Delta_{(x,\lambda)\otimes(y,\mu)}(a) = \Delta'_{(x,\lambda)\otimes(y,\mu)}(a) \cdot R_{1,2} \qquad \forall\, a \in U_q(\widehat{sl(n)}) \tag{2.3}$$

where R satisfies the Yang–Baxter equation

$$R_{1,2} \cdot R_{1,3} \cdot R_{2,3} = R_{2,3} \cdot R_{1,3} \cdot R_{1,2}. \tag{2.4}$$

This is the key to building an integrable model [10].

The solution we have found to (2.3) is in the form

$$\begin{aligned} R_{1,2}(x, y, \lambda, \mu) = {} & \left(y^n - q^2x^n\right) \sum_{i=1}^{n} \lambda^{i-1}\mu^{n-i} e_{i,i} \otimes e_{i,i} + q(y^n - x^n) \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \lambda^{j-1}\mu^{n-i} e_{i,i} \otimes e_{j,j} \\ & + (1-q^2) \sum_{\substack{i,j=1 \\ i<j}}^{n} (\lambda^{i-1}\mu^{n-i}x^{j-i}y^{i-j}y^n e_{i,j} \otimes e_{j,i} \\ & + \lambda^{j-1}\mu^{n-j}x^{i-j}y^{j-i}x^n e_{j,i} \otimes e_{i,j}). \end{aligned} \tag{2.5}$$

We can find a solvable model associated with every solution of the YBE acting on the spaces $C_1^n \otimes C_2^n$ [8]. So, we introduce an one-dimensional lattice with a vector space $V_r \equiv C^n$ at every site. Now, we define an operator per site equal to $R_{1,2}(x, y, \lambda, \lambda_0)$ where the first space C_1^n is an auxiliary space and the second space is the V_r. We call this operator

$$L_r(u,\lambda,\lambda_0) = \sinh\left(\frac{n}{2}u+\gamma\right)\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda_0}\right)^i e_{i,i}\otimes e^r_{i,i} + \sinh\left(\frac{n}{2}u\right)\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\lambda^j}{\lambda_0^i}\right)e_{i,i}\otimes e^r_{j,j}$$

$$+\sinh(\gamma)\sum_{\substack{i,j=1\\ i\neq j}}^{n}\exp\left[\left(i-j-\frac{n}{2}\operatorname{sign}(i-j)\right)u\right]\left(\frac{\lambda}{\lambda_0}\right)^i e_{i,j}\otimes e^r_{j,i} \tag{2.6}$$

where we have made the substitutions

$$\frac{y}{x} = \exp(u) \qquad q = \exp(-\gamma). \tag{2.7}$$

The YBE can now be written as

$$R(u-v,\lambda,\mu)\cdot(L_r(u,\lambda,\lambda_0)\otimes L_r(v,\mu,\lambda_0))$$
$$= (L_r(v,\mu,\lambda_0)\otimes L_r(u,\lambda,\lambda_0))\cdot R(u-v,\lambda,\mu) \tag{2.8}$$

where the $\otimes$ product is in the site space and the $\cdot$ product is in the $A\otimes A$ tensorial space. The operator R in (2.8) is obtained from $R_{1,2}$ in (2.5) by interchanging the indices j and m in every product $e_{i,j}\otimes e_{l,m}$ and the same substitution on its accompanying coefficient. So

$$R(u,\lambda,\mu) = \sinh\left(\frac{n}{2}u+\gamma\right)\sum_{i=1}^{n}\left(\frac{\lambda}{\mu}\right)^i e_{i,i}\otimes e_{i,i} + \sinh\left(\frac{n}{2}u\right)\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\lambda^i}{\mu^j}\right)e_{i,j}\otimes e_{j,i}$$

$$+\sinh(\gamma)\sum_{\substack{i,j=1\\ i\neq j}}^{n}\exp\left[\left(j-i-\frac{n}{2}\operatorname{sign}(j-i)\right)u\right]\left(\frac{\lambda}{\mu}\right)^j e_{i,i}\otimes e_{j,j}. \tag{2.9}$$

With the local operator L_r, we build the monodromy matrix $T(u,\lambda,\lambda_0)$ defined on the auxiliary space A, whose components are operators on the configuration space, i.e. the tensorial product of the site spaces of the lattice

$$T(u,\lambda,\lambda_0)\equiv T(u,\lambda) = L_N(u,\lambda,\lambda_0)\cdot L_{(N-1)}\cdots L_1(u,\lambda,\lambda_0) \tag{2.10}$$

where the $\cdot$ product is understood as before in the auxiliary space.

The monodromy matrix T posesses most of the properties of L_r. The most important is a YBE similar to (2.7):

$$R(u-v,\lambda,\mu)\cdot(T(u,\lambda,\lambda_0)\otimes T(v,\mu,\lambda_0))$$
$$= (T(v,\mu,\lambda_0)\otimes T(u,\lambda,\lambda_0))\cdot R(u-v,\lambda,\mu). \tag{2.11}$$

A consequence of (2.11) is the existence of a commuting family of transfer matrices F, given by the expression

$$F(u,\lambda,\mu) = \mathrm{trace}_{\mathrm{aux}}\, T(u,\lambda,\mu) \tag{2.12}$$

for which

$$[F(u,\lambda,\lambda_0), F(v,\mu,\lambda_0)] = 0 \tag{2.13}$$

as can be proved by taking the trace of (2.11). This implies that differentiating F with respect to the parameters for certain values, we obtain a set of commuting operators. In particular the Hamiltonian, an operator describing the interactions with the two nearest neighbours, is related to the first logarithmic derivative of F. So, we can take the Hamiltonian as

$$H = \frac{2}{n}\sinh\gamma \frac{\partial}{\partial u}\ln(F(u))\bigg|_{\substack{u=0\\ \lambda=\lambda_0}} - \frac{N}{n}\cosh\gamma. \tag{2.14}$$

The derivative with respect to λ

$$Q = -\lambda_0 \frac{\partial}{\partial\lambda}\ln(F)\bigg|_{\substack{u=0\\ \lambda=\lambda_0}} + \left(\frac{n+1}{2}\right)N \tag{2.15}$$

will be a conserved charge. These operators can be expressed as

$$H = \sum_{r=1}^{N-1} h_{r,r+1} \tag{2.16a}$$

$$Q = \sum_{r=1}^{N-1} k_{r,r+1} \tag{2.16b}$$

with

$$\begin{aligned} h_{r,r+1} = \frac{n-1}{n}\cosh(\gamma)\sum_{i=1}^{n} e^r_{i,i}\otimes e^{r+1}_{i,i} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \lambda_0^{(i-j)} e^r_{i,j}\otimes e^{r+1}_{j,i} \\ + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\frac{2(j-i)}{n} - \mathrm{sign}(j-i)\right)\sinh(\gamma) - \frac{\cosh(\gamma)}{n}\right) e^r_{i,i}\otimes e^{r+1}_{j,j} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} k_{r,r+1} &= -\sum_{i=1}^{n} i e^r_{i,i}\otimes e^{r+1}_{i,i} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} j e^r_{i,j}\otimes e^{r+1}_{j,i} + \frac{n+1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} e^r_{i,i}\otimes e^{r+1}_{j,j} \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{n+1}{2} - j\right) e^r_{i,i}\otimes e^{r+1}_{j,j} = I^r \otimes S_z^{(r+1)}. \end{aligned} \tag{2.18}$$

The last expression shows that k is a local operator, the third component of the spin $su(2)$. The operator Q, in view of (2.16), is the sum of the these components and is a conserved quantity.

If we specify for $n = 2$ and $\lambda = \lambda_0 = \exp \mathrm{i}\delta$, we obtain the Heisenberg XXZ model with a Dzyaloshinsky–Moriya interaction [11]:

$$H_{sl(2)} = \frac{1}{2}\sum_{i=1}^{N}\Big(\cos\delta\left(\sigma_x^i\sigma_x^{i+1} + \sigma_y^i\sigma_y^{i+1}\right) + \cosh\gamma\sigma_z^i\sigma_z^{i+1} + \sin\delta\left(\sigma_y^i\sigma_x^{i+1} - \sigma_x^i\sigma_y^{i+1}\right)\Big) \tag{2.19}$$

where the σ are the Pauli matrices. The solution to this model can be found in [12].

For $n = 3$ the Hamiltonian obtained is

$$\begin{aligned} H_{sl(3)} = \frac{1}{2}\sum_{i=1}^{N}\Bigg(&\cos\delta(\lambda_1^i\lambda_1^{i+1} + \lambda_2^i\lambda_2^{i+1} + \lambda_6^i\lambda_6^{i+1} + \lambda_7^i\lambda_7^{i+1}) \\ &+ \sin\delta(\lambda_2^i\lambda_1^{i+1} - \lambda_1^i\lambda_2^{i+1} + \lambda_7^i\lambda_6^{i+1} - \lambda_6^i\lambda_7^{i+1}) \\ &+ \cos(2\delta)\left(\lambda_4^i\lambda_4^{i+1} + \lambda_5^i\lambda_5^{i+1}\right) + \sin(2\delta)\left(\lambda_5^i\lambda_4^{i+1} - \lambda_4^i\lambda_5^{i+1}\right) \\ &+ \cosh\gamma(\lambda_3^i\lambda_3^{i+1} + \lambda_8^i\lambda_8^{i+1}) + \frac{\sinh\gamma}{\sqrt{3}}(\lambda_8^i\lambda_3^{i+1} - \lambda_3^i\lambda_8^{i+1})\Bigg) \end{aligned} \tag{2.20}$$

where we have used the same substitutions as before, and the λ are Gell–Mann matrices.

For $\delta = 0$ these Hamiltonians correspond to the XXZ models and their generalizations to $sl(n)$ [8].

A more specific model is obtained if we perform the substitutions

$$\exp(-\gamma) = \sqrt{pq} \qquad \exp(\mathrm{i}\delta) = \sqrt{\frac{p}{q}} \tag{2.21}$$

we find in this way the models obtained from $SU_{p,q}(n)$. Since that for $n = 2$ we obtain the model of Dasgupta and Chowdhury [5], there must be a relation between $U_r(sl(2))$ and $U_{p,q}(sl(2))$. In fact, the set $\{e, f, k^{\pm 1} \equiv r^{\pm\frac{h}{2}}\}$ of generators of $U_r(sl(2))$ and the set $\{\tilde{e}, \tilde{f}, q^{\pm\frac{h}{2}}, p^{\pm\frac{h}{2}}\}$ of generators of $U_{p,q}(sl(2))$ are $\{\tilde{e}, \tilde{f}, q^{\pm\frac{h}{2}}, p^{\pm\frac{h}{2}}\}$, that respectively satisfy the equations

$$[e, f] = \frac{r^h - r^{-h}}{r - r^{-1}} \qquad [\tilde{e}, \tilde{f}]_{(\frac{p}{q})^{\frac{1}{2}}} \equiv \tilde{e}\tilde{f} - pq^{-1}\tilde{f}\tilde{e} = \frac{q^h - p^{-h}}{q - p^{-1}} \tag{2.22}$$

are related to each other by

$$\tilde{e} = \left(\frac{q}{p}\right)^{\frac{h}{4}} e \qquad \tilde{f} = \left(\frac{q}{p}\right)^{\frac{h}{4}} f. \tag{2.23}$$

In this sense, the models we derived from the quantum group $sl(n)$ with central extension include the model derived by Dasgupta and Chowdhury.

3. Bethe solutions in the $n = 3$ case

The usual method for solving these models is the algebraic Bethe ansatz proposed by Faddeev and his collaborators [1]. For a model with a site space of n components, the method, know as the nested Bethe ansatz [7, 8], is developed in $(n-1)$ steps, every one similar to the Bethe ansatz. In this section we are going to solve the case $n = 3$ and we will show the main features of the model. The generalization to higher values of n of the conserved magnitudes follows immediately.

We start by specifying the monodromy operator (2.10) as

$$T(u,\lambda) = T(u,\lambda,\lambda_0) = \begin{pmatrix} A(u,\lambda) & B_2(u,\lambda) & B_3(u,\lambda) \\ C_2(u,\lambda) & D_{22}(u,\lambda) & D_{23}(u,\lambda) \\ C_3(u,\lambda) & D_{32}(u,\lambda) & D_{33}(u,\lambda) \end{pmatrix}. \tag{3.1}$$

The components are operators in the configuration space of the lattice. Considering

$$B(u,\lambda) = (\, B_2(u,\lambda) \quad B_3(u,\lambda)\,) \qquad D(u,\lambda) = \begin{pmatrix} D_{22}(u,\lambda) & D_{23}(u,\lambda) \\ D_{32}(u,\lambda) & D_{33}(u,\lambda) \end{pmatrix} \tag{3.2}$$

the YBE (2.8) gives the relations

$$B(u,\lambda) \otimes B(v,\mu) = \left[B(v,\mu) \otimes B(u,\lambda)\right] \cdot R^{(2)}(u-v,\lambda,\mu) \tag{3.3a}$$

$$\begin{aligned} A(u,\lambda)B(v,\mu) &= g(v-u)B(v,\mu)A(u,\lambda)s(\lambda) \\ &\quad -B(u,\lambda))A(v,\mu)\cdot \tilde{r}^{(2)}(v-u)s(\lambda) \end{aligned} \tag{3.3b}$$

$$\begin{aligned} D(u,\lambda) \otimes B(v,\mu) &= g(u-v)B(v,\mu) \otimes s(\mu)(D(u,\lambda)\cdot R^{(2)}(u-v,\lambda,\mu)) \\ &\quad -B(u,\lambda) \otimes s(\lambda)(r^{(2)}(u-v)\cdot D(v,\mu)) \end{aligned} \tag{3.3c}$$

g and $h_\pm$ being the functions

$$g(\theta) = \frac{\sinh(\frac{n}{2}\theta+\gamma)}{\sinh(\frac{n}{2}\theta)} \qquad h_\pm = \frac{\sinh(\gamma)\mathrm{e}^{\pm\frac{\theta}{2}}}{\sinh(\frac{n}{2}\theta)} \tag{3.4}$$

with the matrices

$$\begin{aligned} &s(x) = \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} \qquad r^{(2)}(\theta) = \begin{pmatrix} h_-(\theta) & 0 \\ 0 & h_+(\theta) \end{pmatrix} \\ &\tilde{r}^{(2)}(u) = \begin{pmatrix} h_+(\theta) & 0 \\ 0 & h_-(\theta) \end{pmatrix} \qquad R^{(2)}(u,\lambda,\mu) = \begin{pmatrix} \frac{\lambda}{\mu} & 0 & 0 & 0 \\ 0 & \frac{h_-(u)}{g(u)}\frac{\lambda^2}{\mu^2} & \frac{1}{g(u)}\frac{\lambda}{\mu^2} & 0 \\ 0 & \frac{1}{g(u)}\frac{\lambda^2}{\mu} & \frac{h_+(u)}{g(u)}\frac{\lambda}{\mu} & 0 \\ 0 & 0 & 0 & \frac{\lambda^2}{\mu^2} \end{pmatrix}. \end{aligned} \tag{3.5}$$

The state

$$\|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_N \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_1 \tag{3.6}$$

is an eigenstate of the A and $D_{i,j}$ components of T, i.e.

$$A(u,\lambda)\|1\rangle = \left[\sinh(\tfrac{3}{2}u+\gamma)\frac{\lambda}{\lambda_0}\right]^N \|1\rangle = [a(u,\lambda)]^N\|1\rangle \tag{3.7a}$$

$$D_{i,j}(u,\lambda)\|1\rangle = \left[\sinh(\tfrac{3}{2}u+\gamma)\frac{\lambda}{\lambda_0^i}\right]^N \delta_{i,j}\|1\rangle = [d_{i,j}(u,\lambda)]^N\|1\rangle. \tag{3.7b}$$

Now, with the help of relations (3.3), we look for solutions of the equation

$$F(u,\lambda)\Psi_{\lambda_0}(\mu_1,\ldots,\mu_r) = \Lambda(u,\lambda,\lambda_0,\mu_1,\ldots,\mu_r)\Psi_{\lambda_0}(\mu_1,\ldots,\mu_r) \tag{3.8}$$

of the form

$$\Psi_{\lambda_0}(\mu) = \Psi_{\lambda_0}(\mu_1,\ldots,\mu_r) = X_{i_1,\ldots,i_r}B_{i_1}(\mu_1)\otimes\cdots\otimes B_{i_r}(\mu_r)\|1\rangle. \tag{3.9}$$

To begin with, since $\|1\rangle$ is an eigenvector of $A(u)$ and $D_{i,i}$, we apply these operators to Ψ and, by using the commutation relations (3.3), we push the operators A or $D_{i,j}$ through the B to the right. When either A or D reaches $\|1\rangle$ they reproduce this vector again. Since the commutation relations have two terms, this procedure generates many terms. Some of them have the same order of the arguments in the B product; we call them wanted terms. The others have some $B(\mu_j\lambda_0)$ replaced by $B(u,\lambda_0)$ and we call them unwanted terms.

When we apply $F = A + D_{2,2} + D_{3,3}$ to $\Psi(\mu_1,\ldots,\mu_r)$, we collect the unwanted terms and require them to have a vanishing sum. This condition gives us a set of equations for the parameters. The sum of the wanted terms will be required to be proportional to Ψ, providing us with the second part of equation (3.8).

So, the application of $F(u,\lambda)$ on $\Psi_{\lambda_0}(\mu)$ gives the wanted term

$$\begin{aligned}\Bigg[a(u,\lambda)^N\prod_{j=1}^{r}g(\mu_j-u)B_{j_1}(\mu_1,\lambda_0)\otimes\cdots\otimes B_{j_r}(\mu_r,\lambda_0)S^{(r)}(\lambda)X_{j_1,\ldots,j_r}\\ +\prod_{j=1}^{r}g(u-\mu_j)B_{j_1}(\mu_1,\lambda_0)\otimes\cdots\otimes B_{j_r}(\mu_r,\lambda_0)F^{(r)}_{(2)}(u,\mu,\lambda,\lambda_0)X_{j_1,\ldots,j_r}\Bigg]\\ \times\|1\rangle.\end{aligned} \tag{3.10}$$

where

$$S^{(r)}(\lambda) = \underbrace{s(\lambda)\otimes\cdots\otimes s(\lambda)}_{r\text{ times}} \tag{3.11a}$$

$$F^{(r)}_{(2)}(u,\mu,\lambda,\lambda_0) = \lambda_0^r d_{22}^N(u,\lambda)A^{(2)}(u,\mu,\lambda) + \lambda_0^{2r}d_{33}^N(u,\lambda)D^{(2)}(u,\mu,\lambda). \tag{3.11b}$$

Now we impose the cancelation of the unwanted terms. The operators $A^{(2)}$ and $D^{(2)}$ are components of

$$\begin{aligned}T_r^{(2)}(u,\mu,\lambda,\lambda_0)^{i,i_1,\ldots,i_r}_{j,j_1,\ldots,j_r} &= R^{(2)}{}^{a_{r-1},i_r}_{j_r,a_r}(u-\mu_r,\lambda,\lambda_0)\\ &\quad\cdots R^{(2)}{}^{a_1,i_2}_{j_2,a_2}(u-\mu_2,\lambda,\lambda_0)R^{(2)}{}^{j,i_1}_{j_1,a_1}(u-\mu_1,\lambda,\lambda_0)\end{aligned} \tag{3.12}$$

a 2×2 matrix in the components acting on the second and third components of the auxiliary space, which can be written

$$T_r^{(2)}(u,\mu,\lambda,\lambda_0)) = \begin{pmatrix} A_r^{(2)}(u,\mu,\lambda,\lambda_0)) & B_r^{(2)}(u,\mu,\lambda,\lambda_0)) \\ C_r^{(2)}(u,\mu,\lambda,\lambda_0)) & D_r^{(2)}(u,\mu,\lambda,\lambda_0)) \end{pmatrix} \tag{3.13}$$

and its components are operators on the configuration space.

Then, in order for Ψ to be solution of (3.8), we must require that

$$S^{(r)}(\lambda)X = \omega_r(\lambda)X \tag{3.14a}$$

$$F_{(r)}^{(2)}(u,\mu,\lambda,\lambda_0))X = \Lambda_{(r)}^{(2)}(u,\mu,\lambda,\lambda_0))X. \tag{3.14b}$$

The cancelation of the unwanted terms impose the set of equations

$$[a(\mu_k,\lambda_0)]^N \omega_{r-1}(\lambda_0) = \prod_{\substack{j\neq k\\ j=1}}^{r} \frac{g(\mu_k-\mu_j)}{g(\mu_j-\mu_k)} \Lambda_{(r-1)}^{(2)}(\mu_k,\mu,\lambda,\lambda_0) \qquad k=1,\ldots,r. \tag{3.15}$$

The second step is to diagonalize the (3.11b) equation. We apply the same method as in the first step with one unit lower. So, the operator $T_r^{(2)}$ verifies the YBE

$$\begin{aligned} R^{(2)}(u-v,\lambda,\mu)\cdot\left(T_r^{(2)}(u,\mu,\lambda,\lambda_0)\otimes T_r^{(2)}(v,\mu,\mu,\lambda_0)\right) \\ = \left(T_r^{(2)}(v,\mu,\mu,\lambda_0)\otimes T_r^{(2)}(u,\mu,\lambda,\lambda_0)\right)\cdot R^{(2)}(u-v,\lambda,\mu) \end{aligned} \tag{3.16}$$

that gives the relations

$$B^{(2)}(u,\lambda)\cdot B^{(2)}(v,\mu) = \frac{\lambda}{\mu}B^{(2)}(v,\mu)\cdot B^{(2)}(u,\lambda). \tag{3.17a}$$

$$\begin{aligned} A^{(2)}(u,\lambda)\cdot B^{(2)}(v,\mu) &= \lambda g(v-u)B^{(2)}(v,\mu)\cdot A^{(2)}(u,\lambda) \\ &\quad -\lambda h_+(v-u)B^{(2)}(u,\lambda)\cdot A^{(2)}(v,\mu) \end{aligned} \tag{3.17b}$$

$$\begin{aligned} D^{(2)}(u,\lambda)\cdot B^{(2)}(v,\mu) &= \lambda g(u-v)B^{(2)}(v,\mu)\cdot D^{(2)}(u,\lambda) \\ &\quad -\lambda h_-(u-v)B^{(2)}(u,\lambda)\cdot v^{(2)}(v,\mu). \end{aligned} \tag{3.17c}$$

Now we take the state

$$\|1\rangle^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes\cdots\otimes \begin{pmatrix}1\\0\end{pmatrix}_r \tag{3.18}$$

that is an eigenstate of $F^{(2)}$, and we look for eigenstates of the form

$$X = \Psi^{(2)} = B^{(2)}(\rho_1,\mu,\lambda_0)\cdots B^{(2)}(\rho_s,\mu,\lambda_0)\|1\rangle^{(2)} \tag{3.19}$$

that introduce the dependence of the eigenvalues on a new set of parameters $\{\lambda_i\}_{i=1}^{s}$.

Following the same procedures as in the first step, but now in two dimensions,then we find the eigenvalues of $F^{(2)}$ and the conditions that the set of parameters $\{\rho_i\}_{i=1}^{s}$ must satisfy. Then we finally obtain the eigenvalues of F and $F^{(2)}$ as

$$\Lambda(u,\boldsymbol{\mu},\boldsymbol{\rho},\lambda,\lambda_0) = [\sinh(\tfrac{3}{2}u+\gamma)]^N \frac{\lambda^{N+r+s}}{\lambda_0{}^N} \prod_{j=1}^{r} g(\mu_j-u)$$

$$+\prod_{j=1}^{r} g(u-\mu_j)\Lambda^{(2)}_{(r)}(u,\boldsymbol{\mu},\boldsymbol{\rho},\lambda,\lambda_0) \qquad (3.20a)$$

$$\Lambda^{(2)}_{(r)}(u,\boldsymbol{\mu},\boldsymbol{\lambda},\lambda,\lambda_0) = [\sinh(\tfrac{3}{2}u)]^N \frac{\lambda^{N+r+s}}{\lambda_0{}^{2N}}$$

$$\times\left(\prod_{i=1}^{s} g(\rho_i-u) + \frac{1}{\lambda_0{}^N}\prod_{i=1}^{s} g(u-\rho_i)\prod_{j=1}^{r}\frac{1}{g(u-\mu_j)}\right) \qquad (3.20b)$$

and the parameters $\{\mu_i\}_{i=1}^{r}$ and $\{\rho_i\}_{i=1}^{s}$ are solutions of the equations given by the cancelation of the unwanted terms

$$(g(\mu_k))^N \lambda_0{}^N = \prod_{\substack{j=1\\ j\neq k}}^{r} \frac{g(\mu_k-\mu_j)}{g(\mu_j-\mu_k)} \prod_{i=1}^{s} g(\rho_i-\mu_k) \qquad (3.21a)$$

$$\lambda_0{}^N \prod_{j=1}^{r} g(\rho_k-\mu_j) = \prod_{\substack{i=1\\ i\neq k}}^{s} \frac{g(\rho_k-\rho_i)}{g(\rho_i-\rho_k)}. \qquad (3.21b)$$

Every set of solutions for $1 \leqslant s \leqslant r \leqslant N$ of these coupled equations determines an eigenvalue of F.

It must be noted that equations (3.20) and their solutions (3.21) are formally identical to the same equations obtained from a model with a $su_q(n)$ algebra without central extension and twisted boundary conditions [13, 14]. Besides, for $n=2$, a similiarity relation between the R-matrix derived from the $su_q(2)$ and the R-matrix derived from $su_{q,\lambda}(2)$ was shown to exist [14]. That relation can be probably extended to arbitrary n. We will discuss this point in a forthcoming paper.

An analogous set of equations exist for the $\widehat{sl_q(3)}$ model, as can be seen in [7, 8], the difference between that set and (3.21) is the factor $\lambda_0{}^N$ that will modified the solutions for the parameters $\{\mu_j\}_{j=1}^{r}$ and $\{\rho_i\}_{i=1}^{s}$.

The energy spectrum obtained by applying (2.14) to $\Lambda(u,\boldsymbol{\mu},\boldsymbol{\rho},\lambda,\lambda_0)$ is

$$E = \tfrac{2}{3}N\cosh(\gamma) + \sinh(\gamma)\sum_{i=1}^{r}\left(\frac{1}{\tanh(\frac{3}{2}\mu_i)} - \frac{1}{\tanh(\frac{3}{2}\mu_i+\gamma)}\right). \qquad (3.22)$$

As can be seen from the last expression, the energy depends only on the first set of parameters $\{\mu_j\}_{j=1}^{r}$ introduced.

The second operator defined by (2.15) gives the conserved quantity

$$q = -\lambda_0\frac{\partial}{\partial\lambda}\ln\Lambda\bigg|_{\substack{u=0\\ \lambda=\lambda_0}} + 2N = N-(r+s) \qquad (3.23)$$

which is the third component of a chain of spin-1 states of a $SU(2)$ group with $(N-r)$ sites in the state e_1, $(r-s)$ sites in e_2 and s states in e_3.

In conclusion, we can say that the introduction of a coproduct with an element of the centre of the algebras, enables one to find integrable models with new parameters. We have shown that such models are related with the models coming from the algebras $sl_{p,q}(n)$. In addition, we have found the form (2.21) to connect uniparametric models with multiparametric deformations.

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